Analysis IV

(for EL, GM, MX)

Week 6 - 24.03.25

Recall from last time:

If the function f is holomorphic "around" z_0, but has a (possible) singularity at z_0:

- Regular singularity: The Laurent series at $z = z_0$ has no singular part

$$f(z) = c_0 + c_1(z - z_0) + \ldots$$

Equivalent characterizations:

- $\lim_{z \to z_0} f(z)$ exists (in this case: $\lim_{z \to z_0} f(z) = c_0$)
- $f(z)$ extends holomorphically at z_0.

- Pole of order m at z_0:

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \dots ,$$

$$c_{-m} \neq 0$$

In this case: f "blows up" like $\frac{1}{(z-z_0)^m}$ as $z \to z_0$.

Equivalent characterizations:

- $g(z) = (z-z_0)^m f(z)$ has a regular point at z_0, and $g(z_0) \neq 0$. $(g(z_0) = c_{-m})$

- The limit $\lim_{z \to z_0} ((z-z_0)^m f(z))$ exists and is not 0.

So if f has a pole of order m at z_0: I can write it as $f(z) = \frac{g(z)}{(z-z_0)^m}$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

The opposite also true: If $f(z) = \frac{g(z)}{(z-z_0)^m}$, f has a pole of order m at z_0

And $C_{-m} = g(z_0)$.

- **Essential singularity:** Infinitely many non-zero terms in the singular part of the Laurent series.

Example. $f(z) = \dfrac{\sin(z)}{2z^3 + z^4}$, $z_0 = 0$.

$\sin(z)$ vanishes "to first order" at 0,
$2z^3 + z^4$ vanishes "to third order":
We expect to have a pole of order 2.

$$f(z) = \frac{\sin(z)}{2z^3+z^4} = \frac{z - z^3/3! + \dots}{2z^3+z^4} =$$

$$= \frac{1 - z^2/3! + \dots}{2z^2+z^3} = \frac{1}{z^2} \cdot \underbrace{\frac{1 - z^2/3! + \dots}{2+z}}_{g(z):\ \text{Regular at } z_0=0}$$

$$\lim_{z\to 0} \left(z^2 f(z)\right) = \lim_{z\to 0} \frac{1 - z^2/3! + \dots}{2+z} = \frac{1}{2} \neq 0$$

So: pole of order 2.

Example: $f(z) = z \cdot \sin\left(\frac{1}{z}\right)$.

Essential singularity at $z_0 = 0$:

$$\sin\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z}\right)^{2n+1}$$

$$\Rightarrow \quad z \cdot \sin\left(\frac{1}{z}\right) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{2n}}$$

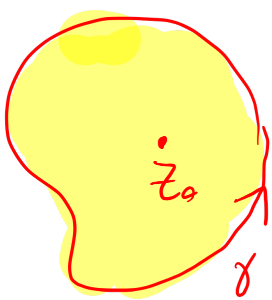
Converges $\forall z \neq 0$.

Note: Don't be tricked by the real valued case: The limit $\lim_{z \to 0} z \cdot \sin\left(\frac{1}{z}\right)$ doesn't exist! The bound $|\sin(w)| \leq 1$ only holds for $w \in \mathbb{R}$.

Residues:

Definition: The coefficient c_{-1} of the term $(z-z_0)^{-1}$ in the Laurent series for f at z_0 is called the residue of f at z_0.

$$\mathrm{Res}_{z_0}(f) := c_{-1}$$



In view of the definition of the coefficients of the Laurent series:

$$c_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz \quad \text{for any}$$

curve γ which is closed, simple, counter-clockwise oriented, contains z_0 in its interior and f is holomorphic on $\mathrm{int}(\gamma)/\{z_0\}$.

Trivial example: If f is holomorphic on a set D containing z_0, then

$$\mathrm{Res}_{z_0}(f) = 0 \qquad \text{(Cauchy's theorem)}$$

Basic example: $\mathrm{Res}_0\left(\frac{1}{z}\right) = 1$

($f(z) = \frac{1}{z}$ already is expressed as a Laurent series at $z_0 = 0$).

Suppose that $f(z)$ has a pole of order m at z_0:

$$f(z) = c_{-m}(z-z_0)^{-m} + \dots + c_{-1}(z-z_0)^{-1} + c_0 + \dots$$

Then: $g(z) = (z-z_0)^m f(z)$

$$= c_{-m} + c_{-m+1}(z-z_0) + \dots$$
$$+ c_{-1}(z-z_0)^{m-1} + \dots$$

is the Taylor series of the regular function g at z_0.

Thus: We can extract the residue using the formula for the Taylor coefficients:

$$\boxed{\begin{aligned} c_{-1} &= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left(g(z_0) \right) \\ &= \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z) \right). \end{aligned}}$$

The above formula is useful if we already know that f has a pole of order m at z_0.

Most useful: When $m=1$. In that case:

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0}\left((z-z_0)\cdot f(z)\right).$$

Example:

$$f(z) = 1/\sin(z), \quad z_0 = 0.$$

Note: $$f(z) = \frac{1}{z - z^3/3! + \dots} = \frac{1}{z}\cdot \underbrace{\frac{1}{1 - z^2/3! + \dots}}_{\text{Regular}}$$

So f: Has a pole of order 1 at $z_0 = 0$

(It would also follow from our more general discussion last time for functions of the form $p(z)/q(z)$).

So: $\operatorname{Res}_0(f) = \lim_{z \to 0} \left(z \cdot f(z) \right) =$

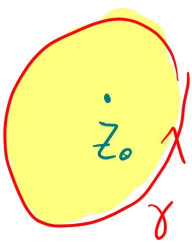
$$= \lim_{z \to 0} \frac{z}{\sin(z)} =$$

$$= \lim_{z \to 0} \frac{z}{z - z^3/3! + \dots} =$$

$$= \lim_{z \to 0} \frac{1}{1 - z^2/3! + \dots} = 1.$$

Residue theorem and applications

Recall: If f is holomorphic "around" z_0, but with a possible singularity at z_0:

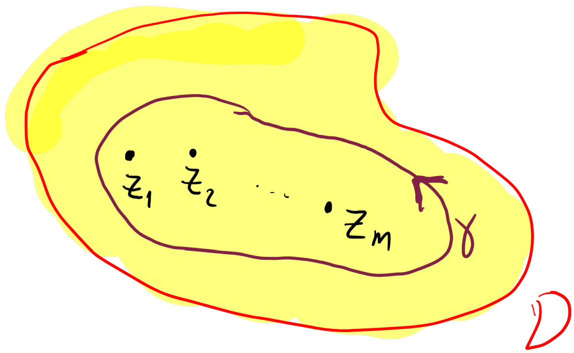


$$\int_\gamma f(z)\,dz = \operatorname{Res}_{z_0}(f)$$

if γ is simple, closed, count.-clockwise contains z_0 and no other singularity of f (so f is holomorphic on the set $\operatorname{int}(\gamma)/\{z_0\}$)

Theorem (Residue theorem)

Suppose that $D \subseteq \mathbb{C}$ is open and simply connected and let γ be a simple, closed curve in D which is counter-clockwise oriented. Let $z_1, z_2, \ldots, z_m \in \operatorname{int}(\gamma)$.



If $f: D \setminus \{z_1, z_2, \ldots, z_m\} \longrightarrow \mathbb{C}$

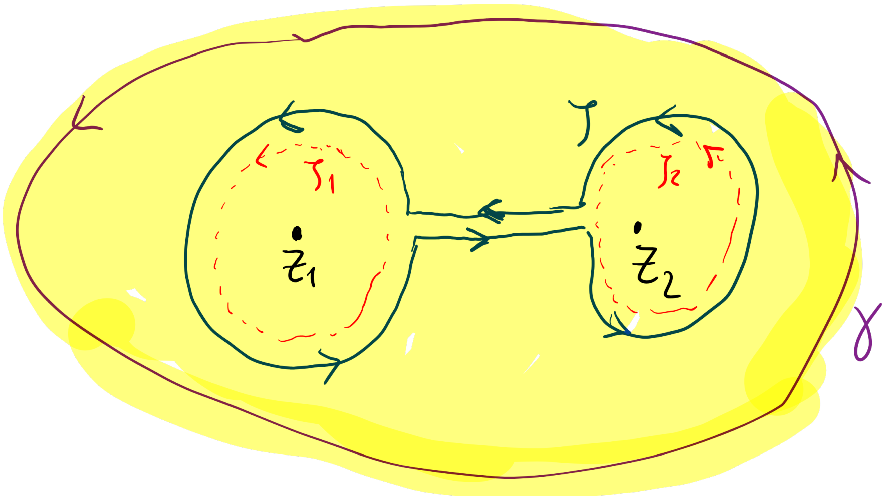
is holomorphic, then

$$\int_\gamma f(z)\, dz = 2\pi i \left(\operatorname{Res}_{z_1}(f) + \ldots + \operatorname{Res}_{z_m}(f) \right).$$

Remark: In order to compute $\int_\gamma f(z)\, dz$, it therefore suffices to compute the residues of f at the singular points in the interior of γ.

Sketch of the proof (by picture):

Assume (without loss of generality) that we only have two singular points:



By the extension of Cauchy's theorem:

$$\int_\gamma f(z)\,dz = \int_{\tilde\gamma} f(z)\,dz$$

When the "bridge" is too narrow: The integral over $\rightleftarrows$ cancels out, leaving us with the two integrals over the smaller loops γ_1 and γ_2 around z_1 and z_2, respectively:

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

$$= 2\pi i \, \mathrm{Res}_{z_1}(f) + 2\pi i \, \mathrm{Res}_{z_2}(f)$$

$\blacksquare$

Example 1

Consider $f(z) = 1/z$, let γ be a simple, closed, counter-clockwise oriented curve in $\mathbb{C}$. Compute $\int_\gamma f(z)\,dz$.

<u>Case 1</u>: $0 \in \text{int}(\gamma)$

Then $\int_\gamma f(z)\,dz = 2\pi i \, \text{Res}_0\left(\frac{1}{z}\right) = 2\pi i$

<u>Case 2</u>: $0 \notin \overline{\text{int}(\gamma)} = \text{int}(\gamma) \cup \gamma$

Then $\int_\gamma f(z)\,dz = 0$.

<u>Case 3</u>: $0 \in \gamma$: In this case, $\int_\gamma f(z)\,dz$ is not well-defined.

[<u>Note</u>: • $\text{int}(\gamma)$: The open set enclosed by γ (so it doesn't contain γ).

• $\overline{\text{int}(\gamma)} = \text{int}(\gamma) \cup \gamma$: The closed set enclosed by γ (so it contains the curve γ).]

Example 2

Let $f : \mathbb{C} \setminus \{0, -1\} \longrightarrow \mathbb{C}$,

$$f(z) = \frac{1}{z \cdot (z+1)}$$

We have two poles of order 1 at $z_1 = 0$ and $z_2 = -1$.

$$\mathrm{Res}_0(f) = \lim_{z \to 0} \left(z \cdot f(z) \right) = \lim_{z \to 0} \frac{1}{z+1} = 1$$

$$\mathrm{Res}_{-1}(f) = \lim_{z \to -1} \left((z+1)\, f(z) \right) = \lim_{z \to -1} \frac{1}{z} = -1$$

Let γ be a simple, closed, counter-clock. oriented curve.

$$\int_\gamma f(z)\, dz = \,?$$

<u>Case A</u>: $0, -1 \notin \overline{\text{int}(\gamma)}$. $\Rightarrow \int_\gamma f(z)dz = 0$.

<u>Case B</u>: $0 \in \text{int}(\gamma)$, $-1 \notin \text{int}(\gamma)$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \,\text{Res}_0(f) = 2\pi i$$

<u>Case C</u>: $0 \notin \overline{\text{int}(\gamma)}$, $-1 \in \text{int}(\gamma)$

$$\Rightarrow \int_\gamma f(z)dz = 2\pi i \,\text{Res}_{-1}(f) = -2\pi i$$

<u>Case D</u>: $0, -1 \in \text{int}(\gamma)$

$$\Rightarrow \int_\gamma f(z)dz = 2\pi i \left(\text{Res}_0(f) + \text{Res}_{-1}(f)\right)$$

$$= 2\pi i \left(1 + (-1)\right) = 0$$

<u>Case E</u>: $0 \in \gamma$ or $-1 \in \gamma$: In this case,

the integral is not well-defined.

Example 3: $f: \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$,

$$f(z) = \frac{1}{z^2} + \frac{2}{z} + \frac{3}{z-1}$$

Pole of order 2 at 0, of order 1 at 1.

$$\mathrm{Res}_1(f) = \lim_{z \to 1} \big((z-1) f(z) \big) = \lim_{z \to 1} \left(\frac{z-1}{z^2} + \frac{2(z-1)}{z} + 3 \right) = 3$$

$$\mathrm{Res}_0(f) = \lim_{z \to 0} \left(\frac{1}{1!} \frac{d}{dz} \left(z^2 f(z) \right) \right) =$$

$$= \lim_{z \to 0} \left(\frac{d}{dz} \left(1 + 2z + \frac{3z^2}{z-1} \right) \right)$$

$$= \lim_{z \to 0} \left(2 + \frac{6z}{z-1} - \frac{3z^2}{(z-1)^2} \right) = 2.$$

(I could have also deduced that by noting that $f(z) = \underbrace{\frac{1}{z^2} + \frac{2}{z}}_{\text{Singular part at } z=0} + \underbrace{\frac{3}{z-1}}_{\text{Regular at } z=0}$

so I can trivially read off c_{-1}).

If $\gamma(t) = 10e^{it}$, $t \in [0, 2\pi]$:

$\operatorname{int}(\gamma)$ contains both poles:

$$\int_\gamma f(z)\,dz = 2\pi i\left(\operatorname{Res}_0(f) + \operatorname{Res}_1(f)\right) = 10\pi i$$

Example 4 $\exp(1/z) = 1 + z^{-1} + \frac{z^{-2}}{2!} + \ldots$

$$\operatorname{Res}_0\left(\exp\left(\tfrac{1}{z}\right)\right) = c_{-1} = 1.$$

Example 5 $\exp(1/z^2) = 1 + z^{-2} + \frac{z^{-4}}{2!} + \dots$

$$\mathrm{Res}_0(f(z)) = 0.$$

Applications

The residue theorem allows us to easily compute integrals over simple closed curves.

Sometimes: I can write integrals over the real line as limits of integrals over closed curves in $\mathbb{C}$.

Example: Compute $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$.

Set $f(z) = \dfrac{1}{z^2+1}$.

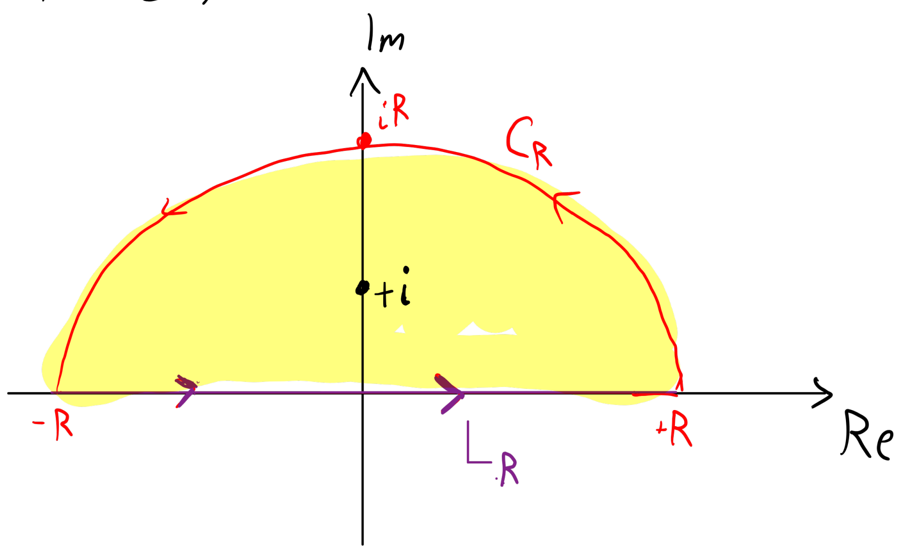
For any $R > 0$: Consider the segment

$$L_R : [-R, R] \longrightarrow \mathbb{C}, \quad t \longmapsto t$$

(segment on the real axis)

and the half circle

$$C_R : [0, \pi] \longrightarrow \mathbb{C}, \quad t \longmapsto R \cdot e^{it}$$



Consider the closed curve $\gamma = L_R \cup C_R$

(boundary of the half-disc of radius R)

The function $f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$

has poles of order 1 at $\pm i$.

From the residue theorem:

$$\int_\gamma f(z)\,dz = 2\pi i \,\mathrm{Res}_i(f)$$ (int(γ) contains only i)

$$\mathrm{Res}_i(f) = \lim_{z\to i}\left((z-i)f(z)\right) = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}$$

So $$\int_\gamma f(z)\,dz = \pi. \quad (1)$$

But $$\int_\gamma f(z)\,dz = \int_{L_R} f(z)\,dz + \int_{C_R} f(z)\,dz.$$

- $$\int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{1}{t^2+1}\,dt$$

$$\xrightarrow{R\to+\infty} \int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx$$

- $$\int_{C_R} f(z)\,dz = \int_0^{\pi} \frac{R\cdot ie^{it}}{1+R^2\cdot e^{2it}}\,dt.$$

We will show that $\int_{C_R} f(z)dz \xrightarrow{R\to+\infty} 0.$

$$\left|\int_{C_R} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{R\cdot ie^{it}}{1+R^2e^{2it}}\,dt\right|$$

$$\leq \int_0^{\pi} \frac{R\cdot|ie^{it}|}{|1+R^2e^{2it}|}\,dt \quad A$$

Note: $|ie^{it}| = 1$ and, when $R \to +\infty$,

$$|1 + R^2 e^{2it}| \approx R^2$$

Since, by the triangle inequality:

$$\underbrace{|R^2 e^{2it}| - 1}_{= R^2 - 1} \leq |R^2 e^{2it} + 1| \leq \underbrace{|R^2 e^{2it}| + 1}_{= R^2 + 1}$$

So: $\quad A \approx \displaystyle\int_0^{\pi} \frac{R}{R^2}\, dt = \frac{\pi}{R} \xrightarrow{R \to +\infty} 0.$

From 1: We deduce that

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx = \pi.$$